A new stationary cylindrically symmetric solution of the Einstein's equations admiting Time machine

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ABSTRACT

In this article a new stationary solution of the Einstein's equations with cosmological constant and Time machine is given. The garavitational field is created by ideal liquid with three massless scalar fields or by ideal liquid with electric-magnetic field and massless scalar field.

1 Introduction

In this article a new stationary solution of the Einstein's equations with cosmological constant and Time machine is given. There are two interpretations of stress-energy tensor for this spacetime. In the first case cosmological constant can be arbitrary and gravitational field is created by ideal liquid with three massless scalar fields. In second case the cosmological constant is nonnegative and matter which creates this gravitational field is an ideal liquid with electric-magnetic field and massless scalar field. In 1937 the first similar solution was found by W.J. van Stockum [1]. But problem of Time machihe is discoursed from 1949, after that cosmological model admiting smooth closed timelike curves was found by the famous logic K. Gödel [2]. He was the first who interpreted similar curves as Time machine.

2 Metric and stress-energy tensor

Let us look the metric

$$ds^2 = \frac{dx^{0^2}}{2\Omega} + 2(x^2dx^1 - x^1dx^2)dx^0 + \Omega(2x^{2^2} - 1)dx^{1^2} + \Omega(2x^{1^2} - 1)dx^{2^2} -$$

$$-4\Omega x^1x^2dx^1dx^2 - dx^{3^2}, \tag{1}$$

where $\Omega = const > 0$. The nonzero components of the Christoffel's symbols are

$$\Gamma^0_{01} = 2x^1,\ \Gamma^0_{02} = 2x^2,\ \Gamma^0_{11} = 8x^1x^2\Omega,\ \Gamma^0_{12} = 4\Omega(x^{2^2} - x^{1^2}),\ \Gamma^0_{22} = -8x^1x^2\Omega,$$

$$\Gamma^1_{02} = -\frac{1}{\Omega},\ \Gamma^1_{12} = -2x^2,\ \Gamma^1_{22} = 4x^1,\ \Gamma^2_{01} = \frac{1}{\Omega},\ \Gamma^2_{11} = 4x^2,\ \Gamma^2_{12} = -2x^1,$$

and nonzero components of the Ricci tensor are

$$R_{00} = \frac{2}{\Omega^2},\ R_{01} = \frac{4x^2}{\Omega},\ R_{02} = \frac{-4x^1}{\Omega},\ R_{11} = 4+8x^{2^2},\ R_{12} = -8x^1x^2,\ R_{22} = 4+8x^{1^2}.$$

Also was calculated scalar curvature $R = -4/\Omega$.

By using the Einstein's equations[1]

$$R_{ik} - \frac{1}{2}g_{ik}R = \kappa T_{ik} + \Lambda g_{ik},$$

[1]The Greek indexes are 1,2,3, and Latin indexes are 0,1,2,3.

we find the following tensor:

$$\kappa T_{ik}+\Lambda g_{ik}=\left[\begin{array}{cccc} \dfrac{3}{\Omega^2} & \dfrac{6x^2}{\Omega} & -\dfrac{6x^1}{\Omega} & 0 \\ \dfrac{6x^2}{\Omega} & 12x^{2^2}+2 & -12x^1x^2 & 0 \\ -\dfrac{6x^1}{\Omega} & -12x^1x^2 & 12x^{1^2}+2 & 0 \\ 0 & 0 & 0 & -\dfrac{2}{\Omega} \end{array}\right]. \tag{2}$$

Note that if we introduce the cylindrical coordinats

$$\left\{\begin{array}{l} x^0=x^0 \\ x^1=r\cos\varphi \\ x^2=r\sin\varphi \\ x^3=x^3 \end{array}\right.$$

then metric is transformed to the expression

$$ds^2=\frac{1}{2\Omega}dx^{0^2}-2r^2dx^0d\varphi-\Omega dr^2+\Omega r^2\left(2r^2-1\right)d\varphi^2-dx^{3^2}.$$

3 The physical interpretations of the stress-energy tensor

3.1 Interpretation 1

In this section we show that geometry of spacetime with metric (1) can be created by ideal liquid, for which we must take

$$\left\{\begin{array}{l} T_{ik}^{(i.liquid)}=(c^2\rho+p)u_iu_k-p\,g_{ik} \\ g^{ik}u_iu_k=1 \end{array}\right. , \tag{3}$$

and for scalar fields

$$T_{ik}^{scalar}=\frac{\partial\varphi}{\partial x^i}\frac{\partial\varphi}{\partial x^k}+\frac{1}{2}g_{ik}(m^2\varphi^2-g^{mn}\frac{\partial\varphi}{\partial x^m}\frac{\partial\varphi}{\partial x^n}), \tag{4}$$

$$-\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^i}(\sqrt{-g}\,g^{ik}\frac{\partial\varphi}{\partial x^k})-m^2\varphi=0. \tag{5}$$

Here (5) is the Klein-Fock's equation.

Let there are three real massless scalar fields and ideal liquid. According to these equations gravitational field will be determinated by equality

$$\kappa T_{ik}+\Lambda g_{ik}=\kappa(c^2\rho+p)u_iu_k+\kappa\frac{\partial\varphi}{\partial x^i}\frac{\partial\varphi}{\partial x^k}+\kappa\frac{\partial\psi}{\partial x^i}\frac{\partial\psi}{\partial x^k}+\kappa\frac{\partial\theta}{\partial x^i}\frac{\partial\theta}{\partial x^k}+g_{ik}\{\Lambda-\kappa p-$$
$$-\frac{\kappa}{2}g^{mn}(\frac{\partial\varphi}{\partial x^m}\frac{\partial\varphi}{\partial x^n}+\frac{\partial\psi}{\partial x^m}\frac{\partial\psi}{\partial x^n}+\frac{\partial\theta}{\partial x^m}\frac{\partial\theta}{\partial x^n})\}. \tag{6}$$

Now we assume that $\varphi=\varphi(x^1)$, $\psi=\psi(x^2)$ and $\theta=\theta(x^3)$, or that only $\partial\varphi/\partial x^1, \partial\psi/\partial x^2$ and $\partial\theta/\partial x^3$ are not equial to zero. The vector

$$u_i=(\pm\frac{1}{\sqrt{2\Omega}},\pm\sqrt{2\Omega}x^2,\mp\sqrt{2\Omega}x^1,0) \tag{7}$$

satisfies to the restriction (3) for 4-velocity. By using these conjectures and (5), and substituting (2) into (6) we obtain

$$\frac{3}{\Omega^2}=\frac{\kappa(c^2\rho+p)}{2\Omega}+\frac{1}{2\Omega}\left(\Lambda-\kappa p+\frac{\kappa}{2\Omega}\left(\frac{\partial\varphi}{\partial x^1}\right)^2+\frac{\kappa}{2\Omega}\left(\frac{\partial\psi}{\partial x^2}\right)^2+\right.$$
$$\left.+\frac{\kappa}{2\Omega}\left(\frac{\partial\theta}{\partial x^3}\right)^2\right)$$
$$\frac{6x^2}{\Omega}=\kappa(c^2\rho+p)x^2+x^2\left(\Lambda-\kappa p+\frac{\kappa}{2\Omega}\left(\frac{\partial\varphi}{\partial x^1}\right)^2+\frac{\kappa}{2\Omega}\left(\frac{\partial\psi}{\partial x^2}\right)^2+\right.$$
$$\left.+\frac{\kappa}{2\Omega}\left(\frac{\partial\theta}{\partial x^3}\right)^2\right)$$
$$-\frac{6x^1}{\Omega}=-\kappa(c^2\rho+p)x^1-x^1\left(\Lambda-\kappa p+\frac{\kappa}{2\Omega}\left(\frac{\partial\varphi}{\partial x^1}\right)^2+\frac{\kappa}{2\Omega}\left(\frac{\partial\psi}{\partial x^2}\right)^2+\right.$$
$$\left.+\frac{\kappa}{2\Omega}\left(\frac{\partial\theta}{\partial x^3}\right)^2\right)$$
$$-12x^1x^2=-2\Omega\,x^1x^2\kappa(c^2\rho+p)-2\Omega\,x^1x^2\left(\Lambda-\kappa p+\frac{\kappa}{2\Omega}\left(\frac{\partial\varphi}{\partial x^1}\right)^2+\right.$$
$$\left.+\frac{\kappa}{2\Omega}\left(\frac{\partial\psi}{\partial x^2}\right)^2+\frac{\kappa}{2\Omega}\left(\frac{\partial\theta}{\partial x^3}\right)^2\right)$$
$$12x^{2\,2}+2=2\Omega x^{2\,2}\kappa(c^2\rho+p)+\kappa\left(\frac{\partial\varphi}{\partial x^1}\right)^2+\Omega(2x^{2\,2}-1)\left(\Lambda-\kappa p+\right.$$
$$\left.+\frac{\kappa}{2\Omega}\left(\frac{\partial\varphi}{\partial x^1}\right)^2+\frac{\kappa}{2\Omega}\left(\frac{\partial\psi}{\partial x^2}\right)^2+\frac{\kappa}{2\Omega}\left(\frac{\partial\theta}{\partial x^3}\right)^2\right)$$

$$12x^{1^2} + 2 = 2\Omega x^{1^2}\kappa(c^2\rho + p) + \kappa\left(\frac{\partial\psi}{\partial x^2}\right)^2 + \Omega(2x^{1^2} - 1)\,(\Lambda - \kappa p +$$
$$+\frac{\kappa}{2\Omega}\left(\frac{\partial\varphi}{\partial x^1}\right)^2 + \frac{\kappa}{2\Omega}\left(\frac{\partial\psi}{\partial x^2}\right)^2 + \frac{\kappa}{2\Omega}\left(\frac{\partial\theta}{\partial x^3}\right)^2\Bigg)$$
$$-\frac{2}{\Omega} = \kappa\left(\frac{\partial\theta}{\partial x^3}\right)^2 - \left(\Lambda - \kappa p + \frac{\kappa}{2\Omega}\left(\frac{\partial\varphi}{\partial x^1}\right)^2 + \frac{\kappa}{2\Omega}\left(\frac{\partial\psi}{\partial x^2}\right)^2 + \frac{\kappa}{2\Omega}\left(\frac{\partial\theta}{\partial x^3}\right)^2\right)$$
$$\frac{\partial}{\partial x^k}\left(g^{1k}\frac{\partial\varphi}{\partial x^1}\right) = \frac{\partial}{\partial x^k}\left(g^{2k}\frac{\partial\psi}{\partial x^2}\right) = \frac{\partial}{\partial x^k}\left(g^{3k}\frac{\partial\theta}{\partial x^3}\right) = 0.$$

The next formulas are directly checked:

$$\begin{cases} \kappa p = \Lambda + \dfrac{2}{\Omega} + \dfrac{\kappa}{2}\left(\dfrac{\partial\theta}{\partial x^3}\right)^2 \\ \kappa c^2\rho = \dfrac{2}{\Omega} - \Lambda - \dfrac{3}{2}\kappa\left(\dfrac{\partial\theta}{\partial x^3}\right)^2 \\ u_i = (\pm\dfrac{1}{\sqrt{2\Omega}}, \pm\sqrt{2\Omega}x^2, \mp\sqrt{2\Omega}x^1, 0) \\ \kappa\left(\dfrac{\partial\varphi}{\partial x^1}\right)^2 = \kappa\left(\dfrac{\partial\psi}{\partial x^2}\right)^2 = 4 + \kappa\Omega\left(\dfrac{\partial\theta}{\partial x^3}\right)^2 \\ \varphi = A_1x^1 + A_2, \psi = B_1x^2 + B_2, \theta = C_1x^3 + C_2 \\ A_1, A_2, B_1, B_2, C_1, C_2 = const. \end{cases} \tag{8}$$

So cosmological constant must change in the following domain

$$-\frac{2}{\Omega} - \frac{\kappa}{2}\left(\frac{\partial\theta}{\partial x^3}\right)^2 \leq \Lambda \leq \frac{2}{\Omega} - \frac{3}{2}\kappa\left(\frac{\partial\theta}{\partial x^3}\right)^2.$$

3.2 Interpretation 2

Above one of the some interpretations of stress-energy tensor was considered. Here we shall attempt to demonstrate another variant of matter. We call attention to the well-known fact that electric-magnetic stress-energy tensor

$$T_{ik}^{el.mag.} = \frac{1}{4\pi}\left(\frac{1}{4}F_{lm}F^{lm}g_{ik} - F_{il}F_k^{\ l}\right). \tag{9}$$

Together with (9) we must take the Maxwell's equations

$$\begin{cases} F_{ik} = \dfrac{\partial A_k}{\partial x^i} - \dfrac{\partial A_i}{\partial x^k} \\ \nabla_k F^{ik} = -\dfrac{4\pi}{c}j^{\,i}, \end{cases} \tag{10}$$

where A_k is 4-potential of this electric-magnetic field and ∇_k is a covariant derivation.

As early we consider ideal liquid and real massless scalar field. Moreover we shall use the electric-magnetic field. And now we show that considered field's system satisfies to (2). We have

$$\kappa T_{ik}+\Lambda g_{ik}=\kappa(c^2\rho+p)u_iu_k+\kappa\frac{\partial\varphi}{\partial x^i}\frac{\partial\varphi}{\partial x^k}-\frac{\kappa}{4\pi}F_{i\,l}F_k^{\;l}+\left\{\frac{\kappa}{16\pi}F_{lm}F^{lm}+\Lambda-\kappa p-\right.$$

$$\left.-\frac{1}{2}\kappa g^{mn}\frac{\partial\varphi}{\partial x^m}\frac{\partial\varphi}{\partial x^n}\right\}g_{ik}. \tag{11}$$

Now we use suggestion (7) and as early consider massless scalar field φ, so that

$$\frac{\partial\varphi}{\partial x^0}=\frac{\partial\varphi}{\partial x^1}=\frac{\partial\varphi}{\partial x^2}=0.$$

Let only $F_{12}\neq 0$. Then

$$\frac{\kappa}{16\pi}F_{lm}F^{lm}=\frac{\kappa}{8\pi\Omega^2}(F_{12})^2,$$

$$-\frac{\kappa}{4\pi}F_{1l}F_1^{\;l}=\frac{\kappa}{4\pi\Omega}\,(F_{12})^2,$$

$$-\frac{\kappa}{4\pi}F_{2l}F_2^{\;l}=\frac{\kappa}{4\pi\Omega}\,(F_{12})^2.$$

By substituting these formulas and (2) into (11) we obtain

$$u_i=(\pm\frac{1}{\sqrt{2\Omega}},\pm\sqrt{2\Omega}x^2,\mp\sqrt{2\Omega}x^1,0)$$

$$\frac{3}{\Omega^2}=\frac{\kappa}{2\Omega}(c^2\rho+p)+\frac{1}{2\Omega}\left\{\frac{\kappa}{8\pi\Omega^2}(F_{12})^2+\Lambda-\kappa p+\frac{1}{2}\kappa\left(\frac{\partial\varphi}{\partial x^3}\right)^2\right\}$$

$$\frac{6x^2}{\Omega}=\kappa(c^2\rho+p)x^2+x^2\left\{\frac{\kappa}{8\pi\Omega^2}(F_{12})^2+\Lambda-\kappa p+\frac{1}{2}\kappa\left(\frac{\partial\varphi}{\partial x^3}\right)^2\right\}$$

$$-\frac{6x^1}{\Omega}=-\kappa(c^2\rho+p)x^1-x^1\left\{\frac{\kappa}{8\pi\Omega^2}(F_{12})^2+\Lambda-\kappa p+\frac{1}{2}\kappa\left(\frac{\partial\varphi}{\partial x^3}\right)^2\right\}$$

$$-12x^1x^2=-2\kappa(c^2\rho+p)\Omega x^1x^2-2\Omega x^1x^2\left\{\frac{\kappa}{8\pi\Omega^2}(F_{12})^2+\Lambda-\kappa p+\frac{1}{2}\kappa\left(\frac{\partial\varphi}{\partial x^3}\right)^2\right\}$$

$$12x^{2^2}+2=2\kappa\Omega(c^2\rho+p)x^{2^2}+\frac{\kappa}{4\pi\Omega}\left(F_{12}\right)^2+\Omega(2x^{2^2}-1)\left\{\frac{\kappa}{8\pi\Omega^2}(F_{12})^2+\Lambda-\right.$$
$$\left.-\kappa p+\frac{1}{2}\kappa\left(\frac{\partial\varphi}{\partial x^3}\right)^2\right\}$$
$$12x^{1^2}+2=2\kappa\Omega(c^2\rho+p)x^{1^2}+\frac{\kappa}{4\pi\Omega}\left(F_{12}\right)^2+\Omega(2x^{1^2}-1)\left\{\frac{\kappa}{8\pi\Omega^2}(F_{12})^2+\Lambda-\right.$$
$$\left.-\kappa p+\frac{1}{2}\kappa\left(\frac{\partial\varphi}{\partial x^3}\right)^2\right\}$$
$$-\frac{2}{\Omega}=\kappa\left(\frac{\partial\varphi}{\partial x^3}\right)^2-\left\{\frac{\kappa}{8\pi\Omega^2}(F_{12})^2+\Lambda-\kappa p+\frac{1}{2}\kappa\left(\frac{\partial\varphi}{\partial x^3}\right)^2\right\}$$
$$\frac{\partial^2\varphi}{\partial x^3\partial x^3}=0.$$

After nondifficult calculations the solution of the system can be written in the form

$$\begin{cases}\Lambda=\kappa p\\ u_i=(\pm\frac{1}{\sqrt{2\Omega}},\pm\sqrt{2\Omega}x^2,\mp\sqrt{2\Omega}x^1,0)\\ (F_{12})^2=\frac{16\pi\Omega}{\kappa}+4\pi\Omega^2\left(\frac{\partial\varphi}{\partial x^3}\right)^2\\ \kappa c^2\rho=\frac{4}{\Omega}-\Lambda-\kappa\left(\frac{\partial\varphi}{\partial x^3}\right)^2\\ \frac{\partial\varphi}{\partial x^3}=const,\frac{\partial\varphi}{\partial x^0}=\frac{\partial\varphi}{\partial x^1}=\frac{\partial\varphi}{\partial x^2}=0\\ 0\le\Lambda\le\frac{4}{\Omega}-\kappa\left(\frac{\partial\varphi}{\partial x^3}\right)^2.\end{cases}\qquad(12)$$

By using (10) we obtain for 4-vector of current

$$j^{\,i}=(\frac{c}{4\pi\Omega}F_{12},0,0,0).$$

Strength and induction of electric and magnetic fields are [3, p.331]

$$E_\alpha=0,\ D^\alpha=\left(\frac{2x^1}{\sqrt{2\Omega^3}}F_{12},\frac{2x^2}{\sqrt{2\Omega^3}}F_{12},0\right),$$

$$H_\alpha=\left(0,0,-\frac{1}{\sqrt{2\Omega^3}}\ F_{12}\right),\quad B^\alpha=\left(0,0,-\frac{1}{\Omega}F_{12}\right).$$

4 Time machine

The metric (1) admits the closed smooth timelike curves. For example we consider the following smooth closed curve

$$L = \{x^0 = const, x^1 = a\sin t, x^2 = a\cos t, x^3 = const\}$$

$$a = const > \frac{1}{\sqrt{2}}.$$

It is timelike in metric (1), that is $g_{ik}dx^i dx^k > 0$:

$$g_{ik}dx^i dx^k = a^2 \left\{ g_{11}\cos^2 t + g_{22}\sin^2 t - 2g_{12}\sin t\cos t \right\} = a^2\Omega(2a^2 - 1) > 0,$$

as $a > 1/\sqrt{2}$.

As it known the distance which Time traveler must go, and his chronometric invariant time are calculated with the help of following formulas [3]:

$$\tau(L) = \frac{1}{c}\oint_L \frac{g_{0i}dx^i}{\sqrt{g_{00}}} = \frac{2\pi a^2\sqrt{2\Omega}}{c},$$

$$l(L) = \oint_L \sqrt{\left(-g_{\alpha\beta} + \frac{g_{0\alpha}g_{0\beta}}{g_{00}}\right)\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}}dt = 2\pi a\sqrt{\Omega}. \qquad (13)$$

So "diameter" of domain which contains a Time machine L, has order $l(L) \sim a\sqrt{\Omega}$. Proper time and time $\tau(l)$ are connected by relation

$$s(L) = \frac{1}{c}\oint_L \sqrt{g_{ik}\frac{dx^i}{dt}\frac{dx^k}{dt}}dt = \frac{1}{c}\oint_L \sqrt{1 - \left(\frac{dl}{d\tau}\right)^2}d\tau.$$

The proper time goes to zero when the velocity of Time machine goes to the velocity of light.

5 Conclusion

In the previous sections we showed that gravitational field, for which geometry of spacetime is discribed by metric (1), can be created by real phisical matter. But it is intresting how the our results and experimental data are in agreement?

At the begining we consider the cosmological solution. As it known in this case density of matter of the Universe is equal to $3 \cdot 10^{-31} g/cm^3$. If we wish

no large distance for Time expedition than $\tau(L)$ and $l(L)$ must be sufficiently small. By using (13) we conclude that under considered assumption we must take small Ω. In result, as our solution discribe the Universe, using (8) and (12), we obtain that scalar field which depend only on third coordinate must be sufficiently large in both interpretations. (But only if in the first interpretation the pressure of liquid is large).

And also we denote that in case of cosmological solution with small Ω in second interpritation the current and the strength and induction of electric and magnetic fields are large.

If we take a solution with large density then considered in previous paragraph scalar field $\theta = \theta(x^3)$ must be small.

Also we notice that in first interpretation we can remove the scalar field θ or ideal liquid, and in second interpretation we can remove the considering scalar field. The investigations of the results without analogous scalar field were discribed in our paper [4]. The evaluations of the nessesary distance and chronometric invariant time for Time travel in model without ideal liquid agree with the evaluations for first interpretation of the stress-energy tensor in [4]. Moreover we denote that there exists interpretation of matter of gravitational field (1) as electric-magnetic field and three real massless scalar field in ideal liquid. The variants of matter which were considered by this article, are particular cases of such interpretation.

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